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Contact Transformations. IV. Contraction of Contact Lie Algebras[1]

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Abstract

Conventional Lie algebra contraction is envisaged as a particular case where the correspondent vector fields defining infinitesimal generators of the transformations are vector fields over a differentiable manifold. More general vector fields over tangent space are considered as infinitesimal generators of contact transformations building up a Lie algebra. A new contraction procedure is defined over such vector fields by means of a Taylor expansion. The most striking feature is that the global structure of Lie algebra remains unchanged, while the individual structure of generators is changed.

1. *General Theory*

Let V be a differentiable manifold of $n + 1$ dimensions with local coordinates $\bar{x}$. ξ is the ring of functions over V. The tangent space $T(V)$ is defined in the usual way as tangent fiber bundles and its sections are vector fields over V. If $f \in \xi$, a vector field is

$$\bar{X} = \bar{f}(\bar{x}) \frac{\partial}{\partial \bar{x}}$$

Let $TT(V)$ be the tangent fiber bundle over $T(V)$. Its sections are vector fields over $T(V)$ of the form

$$\bar{\bar{X}} = \bar{f}(\bar{x}, \dot{\bar{x}}) \frac{\partial}{\partial \bar{x}} + \bar{g}(\bar{x}, \dot{\bar{x}}) \frac{\partial}{\partial \dot{\bar{x}}}$$

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If $x = x(\tau)$ is a curve C over V, the inverse canonical projection $V \to T(V)$ induces in a unique way a curve over $T(V)$. Taking $\bar{x}$: (x_i, x_0) as the $n + 1$ local coordinates over V, we impose

$$\frac{dx_i}{d\tau} = x_i \qquad \text{and} \qquad \frac{dx_0}{d\tau} = k \text{ (const)} \tag{1.1}$$

Therefore, the vector field, valued over C, takes the form

$$\bar{X}_c = \bar{f}(\bar{x}, \dot{\bar{x}}, k) \frac{\partial}{\partial \bar{x}} + \bar{g}(\bar{x}, \dot{\bar{x}}, k) \frac{\partial}{\partial \dot{\bar{x}}/C} \tag{1.2}$$

A *differential equation* over tangent fiber bundles of r order is of the form

$$D(\bar{x}, d\bar{x}/d\tau, d^2\bar{x}/d\tau^2, \ldots, d^r\bar{x}/d\tau^r) = 0 \tag{1.3}$$

We know well the answer to the question: How many and which are the vector fields over $T(V)$ which leave (1.3) unchanged? We suppose that the C curve is a solution of (1.3); thus, the vector fields will be in the form (1.2), and they form a Lie algebra under

$$[\bar{X}_c^{\alpha}; \bar{X}_c^{\beta}] = c_{\alpha\beta}^{\gamma} \bar{X}_{c\gamma} \tag{1.4}$$

(α, β: $1, \ldots, s$) and where s is a dimension of the Lie algebra and $c_{\alpha\beta}^{\gamma}$ are structure constants.

Though the whole analysis that follows can be extended to arbitrary dimension of V, some troubles can be avoided by taking $n + 1 = 2$ in a particular case. Therefore (1.1) becomes[3]

$$\frac{1}{k} v = p = \frac{dx_1}{d\tau} = \dot{x}_1 \qquad \text{and} \qquad \frac{dx_0}{d\tau} = k$$

We expand in powers of p the functions $\bar{f}$ and $\bar{g}$ in (1.2):

$$\bar{X}_c = \left[\bar{f}(\bar{x}, \dot{\bar{x}}) \Big|_{p=0} + \frac{1}{1!} \frac{d\bar{f}}{dp} \Big|_{p=0} p + \frac{1}{2!} \frac{d^2\bar{f}}{dp^2} \Big|_{p=0} p^2 + \cdots \right] \frac{\partial}{\partial \bar{x}}$$
$$+ \left[g(\bar{x}, \dot{\bar{x}}) \Big|_{p=0} + \frac{1}{1!} \frac{dg}{dp} \Big|_{p=0} p + \cdots \right] \frac{\partial}{\partial p} \tag{1.5}$$

Terms in (1.5) from order $1/k^r$ can be suppressed, obtaining a new vector field, up to multiplication on the left by constants. Such a vector field takes the form (i.e., $r - 3$)

[3] We note that if $k = 1$ and $x_0 = t$ we are in the classical mechanics case; if $k = c$ (and again $x_0 = t$), we are in the relativistic case.

$$\bar{X}_C = \left[\bar{f}(\bar{x}, \dot{\bar{x}}) \bigg|_{p=0} = \frac{1}{1!} \frac{d\bar{f}}{dp} \bigg|_{p=0} v + \frac{1}{2!} \frac{d^2\bar{f}}{dp^2} \bigg|_{p=0} v^2 \right] \frac{\partial}{\partial \bar{x}}$$

$$+ \left[g(x, 0) + \frac{1}{1!} \frac{dg}{dp} \bigg|_{p=0} v + \frac{1}{2!} \frac{d^2 g}{dp^2} \bigg|_{p=0} v^2 \right] \frac{\partial}{\partial v} \tag{1.6}$$

defining a new type of contraction over the infinitesimal generators of a contact Lie Algebra. Such generators would be part of another Lie algebra, but could *not* be vector fields over $T(V)$ but only over V, as a result of the expansion. We can see this in a simple example.

2. *Two-Dimensional Example*

The initial motivation for the contraction of Lie algebras was the relation between Poincaré and Galilean Lie algebras, one being the limit of the other (Hill, 1945; Fulton et al., 1962a, b). These groups acting linearly on their manifolds, define infinitesimal generators that are vector fields over such manifolds (not over tangent space). The result is that $\bar{f}$ and $\bar{g}$ are polynomials in x_1, x_0 without explicit dependence on p. Then, contraction changes the structure of Lie algebra (i.e., the $c_{\alpha\beta}$). The corresponding groups are invariance groups of differential equations defining the acceleration in classical and relativistic kinematics.

Now, let us consider the differential equations of such relativistic accelerations to the next order (Segal, 1951; Inonu and Wigner, 1953 and 1954; Saletan, 1961) and let us find their invariance contact group. Contact group in the above-mentioned case is the *same* as pure point transformations. Therefore the infinitesimal generators are vector fields over $T(V)$ as (1.5); and the f's have explicit dependence on p. These infinitesimal generators are (Cerveró, 1973; Boya and Cerveró, 1975a, b)

$$\bar{X}_1 = \frac{p(x_0^2 - x^2)}{2(1-p^2)^{1/2}} \frac{\partial}{\partial x_0} + \frac{(x_0^2 - x^2)}{2(1-p^2)^{1/2}} \frac{\partial}{\partial x} + (1-p^2)^{1/2}(x_0 - xp) \frac{\partial}{\partial p}$$

$$\bar{X}_2 = \frac{p x_0}{(1-p^2)^{1/2}} \frac{\partial}{\partial x_0} + \frac{x}{(1-p^2)^{1/2}} \frac{\partial}{\partial x} + (1-p^2)^{1/2} \frac{\partial}{\partial p} \tag{2.1}$$

$$\bar{X}_3 = \frac{p x}{(1-p^2)^{1/2}} \frac{\partial}{\partial x_0} + \frac{x}{(1-p^2)^{1/2}} \frac{\partial}{\partial x} + p(1-p^2)^{1/2} \frac{\partial}{\partial p}$$

$$\bar{X}_4 = \frac{p}{(1-p^2)} - \frac{\partial}{\partial x_0} + \frac{1}{(1-p^2)^{1/2}} \frac{\partial}{\partial x}$$

plus the conformal group of $E_{1,1}$: i.e., $SO(2, 2)$.[4] Let us concentrate on the four generators of (2.1). We expand in powers of p, extract their c dependence and take in addition the combination

$$\bar{X}_1 - \bar{X}_7, \qquad \bar{X}_2 - \bar{X}_6, \qquad \bar{X}_3 - \bar{X}_5, \qquad \bar{X}_4 - \bar{X}_{10}$$

Letting $c \to \infty$ iff $r > 2$, we get

$$(\bar{X}_1 - \bar{X}_7)^c = Y_1, \qquad (\bar{X}_2 - \bar{X}_6)^c = Y_2, \qquad (\bar{X}_4 - \bar{X}_{10})^c = Y_3$$

and

$$(\bar{X}_3 - \bar{X}_5)^c = -\tfrac{1}{2}(Y_4 + Y_5) \tag{2.2}$$

where $(\bar{X}_2 - \bar{X}_6)^c$ and $(\bar{X}_4 - \bar{X}_{10})^c$ have been rescaled in a different and trivial way (multiplying by c and c^2, respectively).

The Y algebra (invariance contact group of classical accelerations in the considered order) was found in Cerveró (1973). It takes the form

$$Y_1 = (\tfrac{1}{2}vt^2 - tx)\frac{\partial}{\partial t} + (\tfrac{1}{4}v^2t^2 - x^2)\frac{\partial}{\partial x} + (\tfrac{1}{2}v^2t - vx_1)\frac{\partial}{\partial v}$$

$$Y_2 = (vt - x)\frac{\partial}{\partial t} + \tfrac{1}{2}v^2 t\frac{\partial}{\partial x} + \tfrac{1}{2}v^2\frac{\partial}{\partial v}$$

$$Y_3 = v\frac{\partial}{\partial t} + \tfrac{1}{2}v^2\frac{\partial}{\partial x}$$

$$Y_4 = t\frac{\partial}{\partial t} - x\frac{\partial}{\partial x} - 2v\frac{\partial}{\partial v}$$

$$Y_5 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \qquad Y_6 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}$$

$$Y_7 = \frac{t^2}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial v}, \qquad Y = \frac{t^2}{2}\frac{\partial}{\partial t} + xt\frac{\partial}{\partial x} + x\frac{\partial}{\partial v}$$

$$Y_9 = \frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial x}$$

As it has been proved (Boya and Cerveró, 1975a, b) the X and Y Lie algebras have the *same* structure [i.e., isomorphic to $SO(3, 2)$]. In our pro-

[4] $\bar{X}_5 = x_0\frac{\partial}{\partial x_0} + x\frac{\partial}{\partial x}, \qquad \bar{X}_6 = x\frac{\partial}{\partial x_0} + x_0\frac{\partial}{\partial x} + (1 - p^2)\frac{\partial}{\partial p}$

$\bar{X}_7 = xx_0\frac{\partial}{\partial x_0} + \frac{x_0^2 + x^2}{2}\frac{\partial}{\partial x} + x_0(1 - p^2)\frac{\partial}{\partial p}, \qquad \bar{X}_9 = \frac{\partial}{\partial x_0}$

$\bar{X}_8 = \frac{x_0^2 + x^2}{2}\frac{\partial}{\partial x_0} + xx_0\frac{\partial}{\partial x} + x(1 - p^2)\frac{\partial}{\partial p}, \qquad \bar{X}_{10} = \frac{\partial}{\partial x}$

cedure the usual change in the Lie algebra structure is traded for change in the internal structure of infinitesimal generators (i.e., more pure point transformations are found in X algebra than in Y algebra). Complete generalization to more dimensions is under investigation

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